

Bound states of \mathcal{PT} -symmetric separable potentials

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All of the \mathcal{PT} -symmetric potentials that have been studied so far have been local. In this paper nonlocal \mathcal{PT} -symmetric separable potentials of the form $V(x, y) = i\epsilon[U(x)U(y) - U(-x)U(-y)]$, where $U(x)$ is real, are examined. Two specific models are examined. In each case it is shown that there is a parametric region of the coupling strength ϵ for which the \mathcal{PT} symmetry of the Hamiltonian is unbroken and the bound-state energies are real. The critical values of ϵ that bound this region are calculated.

PACS numbers: 11.30.Er, 02.30.Em, 03.65.-w

I. INTRODUCTION

The time-independent Schrödinger equation $H\psi = E\psi$, when written in coordinate space, takes the form of a continuous matrix eigenvalue problem

$$\int dy H(x, y)\psi(y) = E\psi(x). \quad (1)$$

However, all previous studies of non-Hermitian \mathcal{PT} -symmetric quantum-mechanical Hamiltonians have focused on Hamiltonians that in the coordinate representation are diagonal and symmetric. Such Hamiltonians take the form $H = p^2 + V(x)$, where the condition of \mathcal{PT} symmetry is that $V^*(x) = V(-x)$. When expressed as a matrix, H is clearly diagonal,

$$H(x, y) = \partial_x \partial_y \delta(x - y) + V(x) \delta(x - y), \quad (2)$$

and for Hamiltonians of this form the Schrödinger eigenvalue problem (1) is the differential equation

$$-\psi''(x) + v(x)\psi(x) = E\psi(x). \quad (3)$$

The first \mathcal{PT} -symmetric Hamiltonians that were examined in detail belong to the class of diagonal Hamiltonians [1–4]

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \geq 0). \quad (4)$$

The Hamiltonians (4) can have many different spectra depending on the large- $|x|$ boundary conditions that are imposed on the solutions to the corresponding time-independent Schrödinger eigenvalue equation

$$-\psi''(x) + x^2(ix)^\epsilon \psi(x) = E\psi(x). \quad (5)$$

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The boundary conditions on $\psi(x)$ are imposed in Stokes' wedges in the complex- x plane. At the edges of the Stokes' wedges both linearly independent solutions to (5) are oscillatory as $|x| \rightarrow \infty$. However, in the interior of the wedges one solution decays exponentially and the linearly independent solution grows exponentially. The eigenvalues E are determined by requiring that $\psi(x)$ decay exponentially in two nonadjacent wedges. Ordinarily, the eigenvalues are complex, but if the two wedges are \mathcal{PT} -symmetric relative to the imaginary- x axis, then all the eigenvalues are real. (The \mathcal{PT} reflection of the complex number x is the number $-x^*$.)

We emphasize that a Hamiltonian need not be diagonal. Of course, for nondiagonal Hamiltonians it is difficult to solve the Schrödinger eigenvalue problem because it takes the form of the Volterra integro-differential equation (1) rather than the differential equation (3). However, there is an interesting special solvable class of nondiagonal Hamiltonians for which the potential is separable; these Hamiltonians have the form

$$H(x, y) = \partial_x \partial_y \delta(x - y) + U(x)U(y). \quad (6)$$

Potentials of this form are discussed in Ref. [5] and are interesting because they can be used in studies of scattering processes.

The purpose of this paper is to consider the case of complex \mathcal{PT} -symmetric separable potentials. We show that such potentials can have unbroken (and broken) \mathcal{PT} -symmetric regions in which the bound states are real (and complex). In Sec. II we show how to find bound states for real Hermitian separable potentials and in Secs. III and IV we show how to find bound states for complex \mathcal{PT} -symmetric potentials. Finally, in Sec. V we make some brief concluding remarks.

II. BOUND STATES OF HERMITIAN SEPARABLE POTENTIALS

Let us consider the case of a Hamiltonian of the form $H = p^2 + V(x, y)$, where $V(x, y) = gU(x)U(y)$ is a separable potential and g is a coupling strength. Note that if $U(x)$ is real, then H is Hermitian. The Schrödinger eigenvalue equation for this Hamiltonian is

$$-\psi''(x) + gU(x) \int_{-\infty}^{\infty} dy \psi(y)U(y) = E\psi(x), \quad (7)$$

which is a linear integro-differential equation in Volterra form.

In general, the potential $V(x, y)$, may have bound states and scattering states. These states are distinguished by their large- $|x|$ behavior. For a scattering state the wave function $\psi(x)$ does not vanish as $|x| \rightarrow \infty$, but for a bound state $\psi(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Furthermore, if $\psi(x)$ vanishes for large $|x|$, then (7) implies that $U(x)$ must also vanish. Thus, if we wish to solve (7) for a bound state, it is valid to perform a Fourier transform:

$$(E - p^2)\tilde{\psi}(p) = g\tilde{U}(p) \int_{-\infty}^{\infty} dy \psi(y)U(y), \quad (8)$$

where the Fourier transform of $\psi(x)$ is given by

$$\tilde{\psi}(p) \equiv \int_{-\infty}^{\infty} dx e^{ipx} \psi(x) \quad (9)$$

and the inverse Fourier transform of $\tilde{\psi}(p)$ is given by

$$\psi(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{-ipx} \tilde{\psi}(p). \quad (10)$$

Note also that the integral in (8) can be written as a convolution of two Fourier transforms:

$$\int_{-\infty}^{\infty} dy \psi(y) U(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \tilde{U}(-q) \tilde{\psi}(q). \quad (11)$$

To proceed, we multiply (8) by $\tilde{U}(-p) = \tilde{U}^*(p)$, integrate with respect to p , and assume that

$$\alpha = \int_{-\infty}^{\infty} dq \tilde{U}(-q) \tilde{\psi}(q) \neq 0. \quad (12)$$

The result is the secular eigenvalue condition

$$1 = \frac{g}{2\pi} \int_{-\infty}^{\infty} dp \frac{|\tilde{U}(p)|^2}{E - p^2}. \quad (13)$$

The integral in this equation must exist, and it clearly exists if E has a nonzero imaginary part. However, complex energy is associated with scattering, and we are not concerned with scattering states in this paper.

There are two possible ways to have real-energy bound-state solutions to (13). First, E may be negative. In this case the integral exists but it is negative, and this requires that the coupling constant g be negative. While this possibility is viable for the case of Hermitian Hamiltonians, it is not viable for the case of non-Hermitian \mathcal{PT} -symmetric separable Hamiltonians, as we will show in Sec. II. Another way to have a bound state is for $\tilde{U}(p)$ to vanish when $p^2 = E$, where E is a number to be determined consistently, and if this happens, the integral in (13) will exist and there can be a bound state.

Let us construct a potential for which we can solve the one-bound-state problem analytically: We assume for simplicity that $\tilde{U}(p)$ has the form

$$\tilde{U}(p) = e^{-p^2/2} (E - p^2). \quad (14)$$

Then, from evaluating the integral in (13) we get an equation for E in terms of g :

$$E = \frac{1}{2} + \frac{2\sqrt{\pi}}{g}. \quad (15)$$

We also obtain the result that

$$U(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{2\sqrt{\pi}}{g} - \frac{1}{2} + x^2 \right) e^{-x^2/2}. \quad (16)$$

We now verify the consistency of this calculation by showing that α in (12) is nonzero. To do so, we calculate $\tilde{\psi}(p)$ from (8) and find that $\tilde{\psi}(p) = g\alpha e^{-p^2/2}/(2\pi)$, and thus that $\psi(x) = g\alpha(2\pi)^{-3/2} e^{-x^2/2}$. We then evaluate (12) and find that it reduces to the identity $\alpha = \alpha$. Thus, α is an arbitrary normalization constant and the bound-state solution is internally consistent.

III. BOUND STATES OF \mathcal{PT} -SYMMETRIC SEPARABLE POTENTIALS

For the non-Hermitian case we assume that the potential has the \mathcal{PT} -symmetric form $V(x, y) = \epsilon i[W(x, y) - W(-x, -y)]$, where $W(x, y)$ is real. We then further specialize the potential by making it separable and symmetric under the interchange of x and y : $V(x, y) = i\epsilon[U(x)U(y) - U(-x)U(-y)]$, where $U(x)$ is real. This gives the Schrödinger equation

$$-\psi''(x) + i\epsilon \left[U(x) \int_{-\infty}^{\infty} dy \psi(y) U(y) - U(-x) \int_{-\infty}^{\infty} dy \psi(y) U(-y) \right] = E\psi(x). \quad (17)$$

We can rewrite this equation as

$$-\psi''(x) + \frac{i\epsilon}{2\pi} [\alpha U(x) - \beta U(-x)] = E\psi(x), \quad (18)$$

where α and β are expressed as convolutions of the Fourier transform of U and the Fourier transform of ψ :

$$\alpha = \int_{-\infty}^{\infty} dq \tilde{U}(-q) \tilde{\psi}(q), \quad \beta = \int_{-\infty}^{\infty} dq \tilde{U}(q) \tilde{\psi}(q). \quad (19)$$

We assume that α and β are nonzero.

As in the Hermitian case discussed in Sec. II, we solve the Schrödinger equation (18) by taking a Fourier transform:

$$\tilde{\psi}(p) = \frac{i\epsilon}{2\pi} \left[\frac{\tilde{U}(p)}{E - p^2} \alpha - \frac{\tilde{U}(-p)}{E - p^2} \beta \right]. \quad (20)$$

Then, we multiply (20) by $\tilde{U}(-p)$ and integrate to obtain

$$\alpha = \frac{i\epsilon}{2\pi} \left[\alpha \int_{-\infty}^{\infty} dp \frac{|\tilde{U}(p)|^2}{E - p^2} - \beta \int_{-\infty}^{\infty} dp \frac{[\tilde{U}(p)]^2}{E - p^2} \right], \quad (21)$$

and we multiply (20) by $\tilde{U}(p)$ and integrate to obtain

$$\beta = \frac{i\epsilon}{2\pi} \left[\alpha \int_{-\infty}^{\infty} dp \frac{[\tilde{U}(p)]^2}{E - p^2} - \beta \int_{-\infty}^{\infty} dp \frac{|\tilde{U}(p)|^2}{E - p^2} \right]. \quad (22)$$

These two equations have the form

$$\alpha = \frac{i\epsilon}{2\pi} \alpha I_2 - \frac{i\epsilon}{2\pi} \beta I_1, \quad \beta = \frac{i\epsilon}{2\pi} \alpha I_1 - \frac{i\epsilon}{2\pi} \beta I_2, \quad (23)$$

where

$$I_1 = \int_{-\infty}^{\infty} dp \frac{[\tilde{U}(p)]^2}{E - p^2}, \quad I_2 = \int_{-\infty}^{\infty} dp \frac{|\tilde{U}(p)|^2}{E - p^2}. \quad (24)$$

Note that $\tilde{U}(p)$ must be complex [otherwise (23) reduces to triviality $\alpha = \beta = 0$]. It is necessary to assume here that I_1 and I_2 exist. This requires that $\tilde{U}(p)$ vanish at $p^2 = E$. Thus, the eigenvalues of the Hamiltonian must be zeros of $\tilde{U}(p)$. (Other solutions for which

E is complex correspond to scattering states. As stated earlier, in this paper we consider only bound states.)

Because the simultaneous linear equations (23) are homogeneous a solution for α and β exists only if the determinant of the coefficients vanishes. This requirement gives a secular equation that determines the energy E :

$$1 = \frac{\epsilon^2}{4\pi^2} (I_1^2 - I_2^2), \quad (25)$$

which is the analog of (13) for the case of a Hermitian separable potential.

This secular equation has several important properties and consequences. First, like the secular equation for all \mathcal{PT} -symmetric Hamiltonians, it is real if E is real [6]. Note that I_2 in (24) is manifestly real. To see that I_1 in (24) is real, observe that because $U(x)$ is real, $\tilde{U}(p)$ is \mathcal{PT} symmetric; that is, $\tilde{U}^*(p) = \tilde{U}(-p)$. Thus, the change of variable $p \rightarrow -p$ establishes the reality of I_1 . As a consequence, the roots E of this equation are either real or come in complex-conjugate pairs. Second, (25) in conjunction with either of the two equations in (23) implies that

$$|\alpha| = |\beta|. \quad (26)$$

Third, observe that E is a function of the *square* of the coupling constant ϵ so it is independent of the sign of ϵ . This feature is typical of \mathcal{PT} -symmetric Hamiltonians such as $H = p^2 + x^2 + i\epsilon x$ and $H = p^2 + x^2 + \epsilon ix^3$, which have imaginary potentials. Fourth, there is no solution to (25) if E is negative. This is because when $E < 0$ the triangle inequality implies that $I_2^2 \geq I_1^2$, and thus the sign of the right side of (25) is negative for any choice of ϵ . Thus, if there is a bound state, its energy must be positive.

To have a bound state for positive energy E , the function $\tilde{U}(p)$ must vanish at $p^2 = E$. As we did in the Hermitian case discussed in Sec. II, we construct a simple model for which there is just one real root of $\tilde{U}(p)$ at $E = p^2$:

$$\tilde{U}(p) = e^{-p^2/2}(E - p^2)(1 + iap), \quad (27)$$

where a is a *real* constant so that $U(x)$ is a real function, namely,

$$U(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} [E - 1 + (E - 3)ax + x^2 + ax^3]. \quad (28)$$

[Note that if we had chosen this $U(x)$ first, it would not have been obvious that $\tilde{U}(p)$ had a zero.] Then

$$\begin{aligned} I_1 &= \sqrt{\pi} \left(E - \frac{1}{2} - \frac{1}{2}a^2E + \frac{3}{4}a^2 \right), \\ I_2 &= \sqrt{\pi} \left(E - \frac{1}{2} + \frac{1}{2}a^2E - \frac{3}{4}a^2 \right). \end{aligned} \quad (29)$$

Hence,

$$I_1 - I_2 = \frac{1}{2}a^2\sqrt{\pi}(3 - 2E), \quad I_1 + I_2 = \sqrt{\pi}(2E - 1). \quad (30)$$

and the secular equation (25) reduces to the quadratic equation

$$\frac{8\pi}{a^2\epsilon^2} + (2E - 1)(2E - 3) = 0, \quad (31)$$

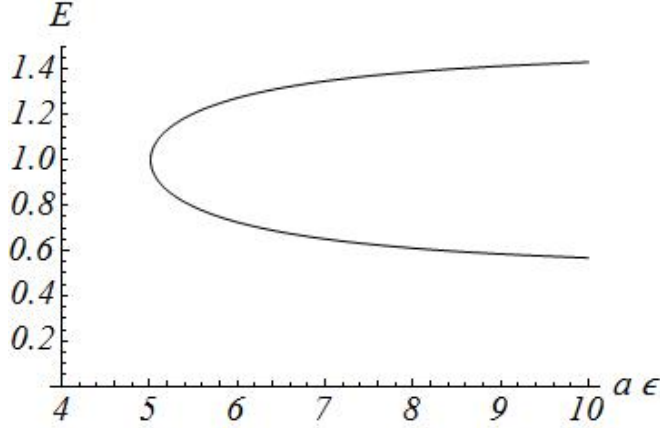


FIG. 1: The bound-state energy E in (32) plotted as a function of $a\epsilon$. The region of unbroken \mathcal{PT} symmetry is $a\epsilon \geq \sqrt{8\pi} = 5.013\dots$. In this region the Hamiltonian has only one bound state, but there are two possible allowed values for the energy of this bound-state. As one can see from (32), the two allowed values of the energy approach the asymptotic limits $\frac{3}{2}$ and $\frac{1}{2}$ as $a\epsilon \rightarrow \infty$.

whose roots are

$$E = 1 \pm \frac{1}{2} \sqrt{1 - \frac{8\pi}{a^2\epsilon^2}}. \quad (32)$$

This equation gives a condition on the coupling constant ϵ for the reality of the bound-state energy E :

$$\epsilon \geq \frac{\sqrt{8\pi}}{a}. \quad (33)$$

The interpretation of this inequality is straightforward. Even though $U(x) \rightarrow 0$ for large $|x|$ it is still possible for there to be a bound state if ϵ is large enough. However, when ϵ is smaller than a critical value, the bound state disappears (its energy becomes complex). This corresponds to going from a \mathcal{PT} -unbroken region to a \mathcal{PT} -broken region. A plot of E as a function of $a\epsilon$ is given in Fig. 1. Note that the Hamiltonian has only *one* bound state; however, there are two allowed values for the energy of this bound state.

Finally, we must verify that apart from a constant multiplicative phase [which is arbitrary because the Schrödinger equation (17) is linear] the eigenfunction $\psi(x)$ is \mathcal{PT} symmetric. To do so, we calculate $\tilde{\psi}(p)$ and verify that apart from a constant multiplicative phase the function $\tilde{\psi}(p)$ is a real function of p . To obtain $\tilde{\psi}(p)$ we substitute (27) into (20) and obtain

$$\begin{aligned} \tilde{\psi}(p) &= \frac{\epsilon}{2\pi} e^{-p^2/2} [i(\alpha - \beta) - ap(\alpha + \beta)] \\ &= \frac{\epsilon}{2\pi} (\alpha + \beta) e^{-p^2/2} \left(i \frac{\alpha - \beta}{\alpha + \beta} - ap \right). \end{aligned} \quad (34)$$

Next, we add the two equations in (23) and immediately obtain the result that the ratio $(\alpha - \beta)/(\alpha + \beta) = -2\pi i/(I_1 + I_2)$ is imaginary. This verifies that up to an arbitrary multiplicative phase, $\psi(x)$ is \mathcal{PT} symmetric; it becomes \mathcal{PT} symmetric when $\alpha + \beta$ is chosen to be real, so that $\alpha = \beta^*$. If the coupling constant ϵ lies below the critical point

$\epsilon < 2\pi\sqrt{2}/a$, then E becomes complex; consequently, I_1 and I_2 are not real and $\psi(x)$ is not \mathcal{PT} symmetric.

IV. SEPARATED \mathcal{PT} -SYMMETRIC POTENTIAL HAVING TWO BOUND STATES

Although it becomes technically complicated, it is straightforward to generalize the discussion of Sec. III to the case in which the \mathcal{PT} -symmetric separated potential has more than one bound state. In this section we consider the case in which there are two bound states of energy E_1 and E_2 for the Schrödinger equation (17).

The bound-state eigenfunctions are labeled $\psi_k(x)$ ($k = 1, 2$), and in momentum space they have the same general form as in (20):

$$\tilde{\psi}_k(p) = \frac{i\epsilon}{2\pi} \left[\frac{\tilde{U}(p)}{E_k - p^2} \alpha_k - \frac{\tilde{U}(-p)}{E_k - p^2} \beta_k \right] \quad (k = 1, 2), \quad (35)$$

where

$$\alpha_k = \int_{-\infty}^{\infty} dq \tilde{U}(-q) \tilde{\psi}_k(q), \quad \beta_k = \int_{-\infty}^{\infty} dq \tilde{U}(q) \tilde{\psi}_k(q) \quad (k = 1, 2). \quad (36)$$

Equations (21) and (22) generalize to

$$\begin{aligned} \alpha_k &= \frac{i\epsilon}{2\pi} \left[\alpha_k \int_{-\infty}^{\infty} dp \frac{|\tilde{U}(p)|^2}{E_k - p^2} - \beta_k \int_{-\infty}^{\infty} dp \frac{[\tilde{U}(p)]^2}{E_k - p^2} \right] \quad (k = 1, 2), \\ \beta_k &= \frac{i\epsilon}{2\pi} \left[\alpha_k \int_{-\infty}^{\infty} dp \frac{[\tilde{U}(p)]^2}{E_k - p^2} - \beta_k \int_{-\infty}^{\infty} dp \frac{|\tilde{U}(p)|^2}{E_k - p^2} \right] \quad (k = 1, 2). \end{aligned} \quad (37)$$

We then extend (27) so that $\tilde{U}(p)$ has two real zeros instead of one:

$$\tilde{U}(p) = e^{-p^2/2} (E_1 - p^2)(E_2 - p^2)(1 + iap). \quad (38)$$

The secular equations satisfied by the energies E_1 and E_2 are a rather complicated generalization of (31):

$$\begin{aligned} \frac{128\pi}{a^2\epsilon^2} + P(E_1, E_2)Q(E_1, E_2) &= 0, \\ \frac{128\pi}{a^2\epsilon^2} + P(E_2, E_1)Q(E_2, E_1) &= 0, \end{aligned} \quad (39)$$

where

$$\begin{aligned} P(E_1, E_2) &= -15 + 6E_1 + 12E_2 - 8E_1E_2 - 4E_2^2 + 8E_1E_2^2, \\ Q(E_1, E_2) &= -105 + 30E_1 + 60E_2 - 24E_1E_2 - 12E_2^2 + 8E_1E_2^2. \end{aligned} \quad (40)$$

By solving simultaneously the equations (39) and (40), we obtain the allowed bound state energies. The numerical solution of these equations is plotted in Fig. 2. There is a critical point at $a\epsilon = 1.09$; below this value there are no real solutions. Bound states first appear when $a\epsilon$ exceeds this critical value and there are two cases: (i) There can be just one bound

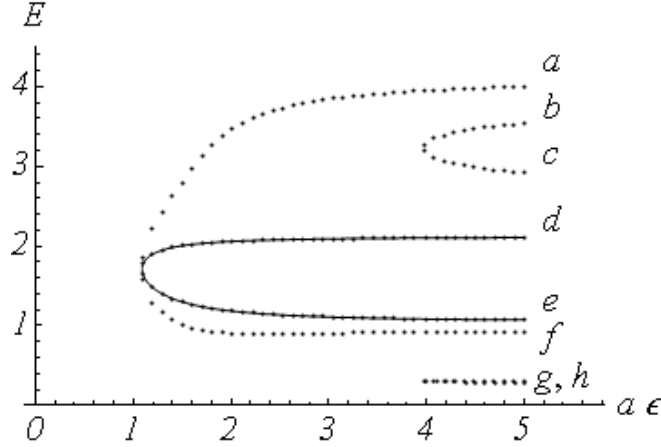


FIG. 2: Numerical solution to the simultaneous equations (39) and (40). Bound-state energies are graphed as functions of $a\epsilon$. The region of unbroken \mathcal{PT} symmetry is $a\epsilon \geq 1.09$, and bound states appear as soon as $a\epsilon$ exceeds this critical value. A second critical point is at $a\epsilon = 3.90$, at which four new solutions appear. There are two cases: In the first case there is only one bound state whose energy may lie on the curve d or on the curve e . In the second case there are two bound-state energies, which may lie on the curves a and f , or on b and g , or on c and h . The two curves g and h are too close to be resolved in this figure and are thus shown in detail in Fig. 3. As $a\epsilon \rightarrow \infty$, the curves $a - h$ approach the asymptotic values 4.081, 3.742, 2.725, 2.115, 1.054, 0.919, 0.296, and 0.275.

state if $E_1 = E_2$, and in this case there are two possible values for this energy and these are indicated in Fig. 2 by the curves consisting of connected dots and labeled d and e . (ii) There can be two distinct bound states; these are indicated by the sequences of dots labeled a and f . If E_1 lies on the curve a , then E_2 lies on f , and *vice versa*. It is interesting that there is a second critical point at $a\epsilon = 3.90$ and when $a\epsilon$ exceeds this value four new possible energies appear; these energies lie on the curves labeled by b , c , g , and h . The possible bound-state energies E_1 and E_2 are only allowed to lie on the pairs of curves a and f , or b and g , or c and h . As $a\epsilon \rightarrow \infty$, the energies on the curves $a - h$ approach the asymptotic values 4.081, 3.742, 2.725, 2.115, 1.054, 0.919, 0.296, and 0.275. The energies on the curves g and h are too close together to be resolved in Fig. 2, so we have included a separate figure Fig. 3 to show their dependence on $a\epsilon$.

V. COMMENTS AND DISCUSSION

We have shown in this paper that it is easy to construct non-Hermitian \mathcal{PT} -symmetric separable potentials and, even though such potentials are nonlocal, it is still possible to find the secular equation that determines the bound-state energies. As is the case for any \mathcal{PT} -symmetric potential, the secular equation is real. If one solves the secular equation, one finds a result that is typical of \mathcal{PT} -symmetric theories; namely, that the coupling constant lies in one of two regions, which are separated by a critical value: On one side of the critical value (the unbroken region of unbroken \mathcal{PT} symmetry) the energies are real, but on the other side of the critical point (the region of broken \mathcal{PT} symmetry) the energies are complex. This \mathcal{PT}

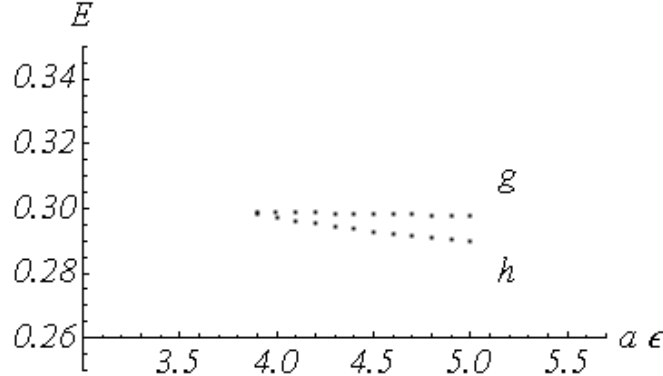


FIG. 3: Blow-up of the curves g and h from Fig. 2.

phase transition has been observed experimentally in \mathcal{PT} -symmetric optical models [7, 8].

Based on the structure and behavior of the models we have constructed in this paper, it is evident that we can construct \mathcal{PT} -symmetric separable potentials for which there are as many bound states as we wish, and in the unbroken \mathcal{PT} -symmetric region all of the bound states will have positive energies.

For the models discussed in this paper the potentials vanish exponentially rapidly as $|x| \rightarrow \infty$. Thus, for large $|x|$ the Hamiltonian becomes the free Hamiltonian $H_0 = p^2$, whose solutions are plane waves. Thus, in addition to bound states, there will be scattering states for all positive energies. The energy of a scattering state will be complex, with the sign of the imaginary part of the energy being associated with incoming- or outgoing-wave boundary conditions. The models we have studied in this paper are interesting because the point spectrum of bound states is embedded in the continuum of scattering states [9–11].

Acknowledgments

CMB is supported by a grant from the U.S. Department of Energy.

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